Given a function $f : I \to \mathbb{R}$ and $c \in I$, we say that $f$ is **differentiable** at $c$ if the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

exists. If it exists, we define the **derivative** of $f$ at $c$, written $f'(c)$, to be the above limit. We say $f$ is **differentiable** if it is differentiable at all $c \in I$. In this case, we can define the **derivative function** (or simply **derivative**) $f' : I \to \mathbb{R}$ by $x \mapsto f'(x)$.

Recall some properties of the derivative we have shown:

- The derivative is linear: if $f$ and $g$ are both differentiable at $c$, then so is $f + g$, and $(f + g)'(c) = f'(c) + g'(c)$. If $f$ is differentiable at $c$ and $\alpha \in \mathbb{R}$, then so is $\alpha f$, and $(\alpha f)'(c) = \alpha f'(c)$.
- Power rule: if $f : I \to \mathbb{R}$ is defined by $f(x) = x^n$ with $n \in \mathbb{N}$, then $f$ is differentiable and
$$f'(x) = nx^{n-1}.$$
- Product rule: if $f$ and $g$ are both differentiable at $c$, then so is $fg$, with
$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$
- Quotient rule: if $f$ and $g$ are both differentiable at $c$, and $g(c) \neq 0$, then so is $f/g$, with
$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

**Problem 1**

Find the derivative of the following functions.

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 - 157x + 10^{48}$.
2. $f : \mathbb{R} \to \mathbb{R}, f(x) = \dfrac{x^3 - 157x + 10^{48}}{2 + x^4}$.
3. $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}, f(x) = \dfrac{x^3 - \frac{157}{x-1} + 10^{48}}{2 + x^4}$.

**Problem 2**

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{x}$. Show that

$$f'(x) = \frac{1}{2\sqrt{x}}$$

for all $x$ using the limit definition of the derivative.

**Problem 3**

Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = \frac{1}{x^n}$ with $n \in \mathbb{N}$. Show that

$$f'(x) = -\frac{n}{x^{n+1}}$$

for all $x \neq 0$ using the limit definition of the derivative.
*Hint:* $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$.

**Problem 4 (Generalized Product Rule)**
Suppose $f_1, f_2, \ldots, f_n$ are functions that are all differentiable at $c$. Let $f$ be the product of all $f_i$:

$$f = \prod_{i=1}^{n} f_i.$$

Show that

$$f'(c) = \sum_{i=1}^{n} \left( f_i'(c) \prod_{j=1, j\neq i}^{n} f_j(c) \right).$$

What does this say if $n = 2$? *Hint: Use induction and the product rule.*

**Problem 5**
Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ x^2 & x \in \mathbb{Q}. \end{cases}$$

On which points (if any) is $g$ differentiable? Where is $g$ non-differentiable?

**Problem 6**
Let $c > 0$. Find the area of the triangle bounded between the $x$-axis, the $y$-axis, and the line tangent to the curve $y = \frac{1}{x}$ at $c$.